Topology and its Applications 156 (2009) 956–962

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Group action on $\mathbb{R} \times \mathbb{Q}$ and fine group topologies ☆

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ARTICLE INFO

Article history:
Received 17 December 2007
Received in revised form 20 November 2008

MSC:
54C35
57S05
54H99

Keywords:
Evaluation function
Admissibility
Weil uniformity
Uniformity of uniform convergence
Uniform topology
Homeomorphism group
Topological group topologies
Rim-compact spaces
Group action
Fine uniform topology
Fine uniform topology w.r.t. a class of metrics
Fine group topology with respect to a class of metrics
Whitney topology
Open-cover topology
Limitation topology

ABSTRACT

Let X be a Tychonoff space, $\mathcal{H}(X)$ the group of all self-homeomorphisms of X and $e:(f,x) \in \mathcal{H}(X) \times X \to f(x) \in X$ the evaluation map. Let $\mathcal{L}_{\mathcal{H}}(X)$ be the upper-semilattice of all group topologies on $\mathcal{H}(X)$ with the additional property that the evaluation map is continuous (ordered by inclusion). The existence of a least element in $\mathcal{L}_{\mathcal{H}}(X)$ has been proven for T_2 locally compact spaces, for T_2 rim-compact and locally connected spaces and for products of T_2 zero-dimensional spaces satisfying the property: *any two non-empty clopen subspaces are homeomorphic*. We show that X being rim-compact is not a necessary condition in order for $\mathcal{L}_{\mathcal{H}}(X)$ to have a least element. Let $\mathbb{R}$ and $\mathbb{Q}$ be the sets of the real and rational numbers respectively, both carrying the Euclidean topology. It is known that $\mathbb{R} \times \mathbb{Q}$ is not rim-compact. We prove that $\mathcal{L}_{\mathcal{H}}(\mathbb{R} \times \mathbb{Q})$ admits a least element.

1. Introduction

Let X be a Tychonoff space, $\mathcal{H}(X)$ the group of all self-homeomorphisms of X with the usual composition and $e:(f,x) \in \mathcal{H}(X) \times X \to f(x) \in X$ the evaluation map. We call a *group topology* on $\mathcal{H}(X)$ any topology on $\mathcal{H}(X)$ which makes $\mathcal{H}(X)$ a topological group, see [5]. We call *admissible* any topology on $\mathcal{H}(X)$ which makes the evaluation map continuous, see [1]. Of course, any admissible group topology on $\mathcal{H}(X)$ makes the evaluation map a group action of $\mathcal{H}(X)$ on X, see [16]. We denote by $\mathcal{L}_{\mathcal{H}}(X)$ the set of all admissible group topologies on $\mathcal{H}(X)$ ordered by the usual inclusion. Since every topology finer than an admissible one is in turn admissible and the join of subsets of group topologies is again a group topology, therefore $\mathcal{L}_H(X)$ is a complete upper-semilattice. Obviously, the discrete topology is in $\mathcal{L}_H(X)$ and is indeed the greatest element. As a consequence, $\mathcal{L}_H(X)$ having the least element is equivalent to $\mathcal{L}_H(X)$ being a complete lattice.

☆ This work has been supported by Fondi 60% MIUR, Italy.
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0166-8641/$ – see front matter
doi:10.1016/j.topol.2008.11.013

In this paper we consider the question of when $\mathcal{L}_H(X)$ has a least element for a non-compact space X. This area of research originated from the early work of G. Birkoff [3], who considered compact metric spaces. Later R. Arens [1], considered the class of T_2 locally compact spaces. He proved that if X is locally compact T_2, then the least element, which he called the g-topology, is that topology generated by the collection of all sets of the form

$$[C, W] = \{f \in \mathcal{H}(X):\ f(C) \subseteq W\},$$

where C is a closed subset of X, W is an open subset of X and C or $X \setminus W$ is compact. In addition, if X is also locally connected, then the g-topology reduces to the compact-open topology [8,11].

Recall that a space X is called *rim-compact* iff every point in X has arbitrarily small neighbourhoods with compact boundary. The author considered rim-compact spaces in [6,7] and proved that $\mathcal{L}_{\mathcal{H}}(X)$ has a least element if X is rim-compact, T_2 and locally connected, or if X is the product of T_2 zero-dimensional, rim-compact spaces each satisfying the property that any two non-empty clopen subspaces are homeomorphic.

In all of these results, the least element in $\mathcal{L}_{\mathcal{H}}(X)$ was constructed as the uniform topology induced by the unique totally bounded uniformity associated with a T_2-compactification of X, to which every self-homeomorphism of X continuously extends. In particular, such well-known T_2-compactifications as the one-point compactification, the Freudenthal compactification, the Stone–Čech compactification are of this sort.

As rim-compactness is a weak and peripherical compactness property, one might think any further relaxation as impossible. But, in this paper, we show that X being rim-compact is not a necessary condition in order for a least admissible group topology to exist.

We use the standard notation that $\mathbb{R}$ denotes the set of real numbers and $\mathbb{Q}$ denotes the set of rational numbers each with their usual topology. It is known that $\mathbb{R} \times \mathbb{Q}$ with the product topology is not rim-compact [12]. Our main result in this paper is the following:

Theorem 1.1. *The set $\mathcal{L}_{\mathcal{H}(\mathbb{R}\times\mathbb{Q})}$ has a least element. Indeed, the least element is the fine group topology on $\mathcal{H}(\mathbb{R} \times \mathbb{Q})$ generated by the class of all metrics on $\mathbb{R} \times \mathbb{Q}$ of the type $d_1 \times d_2$, where d_1 is the stereographic metric on $\mathbb{R}$ and d_2 runs through all totally bounded metrics compatible with the usual topology on $\mathbb{Q}$.*

In order to prove this result, we introduce a new method, other than the compact extension procedure, to produce admissible group topologies on $\mathcal{H}(X)$ and its subgroups. The motivation for the new method comes from the construction of the least element of $\mathcal{L}_{\mathcal{H}(\mathbb{Q})}$ in [6] and the presentation in [2] of the fine uniform topology. These point to using the suprema of uniform topologies derived from a given class of compatible metrics on X which is a very natural generalisation of the fine uniform topology and leads to what we call the fine group topology associated with the given class of metrics.

The proof of Theorem 1.1 uses a natural embedding of $\mathcal{H}(\mathbb{R} \times \mathbb{Q})$ in $C(\mathbb{R} \times \mathbb{Q}, \mathbb{R}) \times \mathcal{H}(\mathbb{Q})$, where $C(\mathbb{R} \times \mathbb{Q}, \mathbb{R})$ is the set of all continuous functions from $\mathbb{R} \times \mathbb{Q}$ to the reals.

The homeomorphism group $\mathcal{H}(X)$ can be equipped with several interesting admissible group topologies. In Section 2 we summarise properties of uniform topologies such as the fine uniform topology, and in the metric case, of the limitation topology and of the Whitney topology (also called the fine topology). We also show that, whenever the Whitney topology is a group topology, then it agrees with the fine uniform topology.

In Section 3 we consider $\mathcal{H}(\mathbb{R} \times \mathbb{Q})$. We split each self-homeomorphism of $\mathbb{R} \times \mathbb{Q}$ into two natural halves, using the natural embedding mentioned above. This splitting results in a considerable simplification in dealing with $\mathcal{H}(\mathbb{R} \times \mathbb{Q})$.

In Section 4 we introduce the concepts of fine uniform topology associated with a given class of metrics and the fine group topology associated with a given class of metrics.

In Section 5 we prove our main result.

2. Background

In this section we review several well-known topologies on $\mathcal{H}(X)$ which motivate the topology we use in our main result.

The definitions, terminology and results quoted below are drawn from [5,9,16,18]. We recall some specific topologies on $\mathcal{H}(X)$, which play a role in the sequel and we differentiate them by their method of construction. We also show that whenever the Whitney topology is a group topology, then it agrees with the finest uniform topology.

- *How uniformities on X yield uniformities on $\mathcal{H}(X)$ (see* [10,18]).

Let X stand for a Tychonoff space. Every Weil uniformity $\mathcal{U}$ compatible with X induces on $\mathcal{H}(X)$ the *uniformity of uniform convergence w.r.t. $\mathcal{U}$*, which admits as basic diagonal neighbourhoods the sets:

$$\hat{U} := \{(f, g) \in \mathcal{H}(X) \times \mathcal{H}(X):\ (f(x), g(x)) \in U,\ \forall x \in X\}$$

as U runs through all diagonal neighbourhoods in $\mathcal{U}$. The uniformity of uniform convergence w.r.t. $\mathcal{U}$ on $\mathcal{H}(X)$ generates in turn the *uniform topology* or the *topology of uniform convergence w.r.t. $\mathcal{U}$*, that we will denote as τ_U. Whenever the uniformity $\mathcal{U}$ is metrisable and d is a bounded metric compatible with it, then the uniform topology τ_U is just the topology of the *supremum metric* $\hat{d}$, which is determined from d by the usual formula:

$$\hat{d}(f,g) := \sup\{d(f(x),g(x)):\ x \in X\}.$$

The uniform topology induced on $\mathcal{H}(X)$ by the finest uniformity compatible with X is usually referred to as the *fine uniform topology on* $\mathcal{H}(X)$. Following the notation in [14], we will denote it as τ_f.

We now summarise a number of basic facts. For a Weil uniformisable space X, every uniform topology on $\mathcal{H}(X)$ is admissible, see [10, 2.6.C]. Furthermore, every uniform topology implies continuity of the inverse map of $\mathcal{H}(X)$ at i, and continuity of the product of $\mathcal{H}(X)$ at (i,i), where i is the identity map of X. But, in general, a uniform topology is not a group topology. Also, if X is a metrisable separable space, thus having compatible totally bounded metrics, then the uniform topology on $\mathcal{H}(X)$ induced by the Čech uniformity of X, which is also the finest totally bounded uniformity compatible with X, is the supremum of all uniform topologies derived from totally bounded metrics compatible with X. Finally, in the metric case the fine uniform topology, τ_f, is the supremum of all uniform topologies derived from metrics compatible with X, [15].

- *Using real functions in the metric case*: *the Whitney topology* (*see* [13]).

Let (X,d) stand for a metric space. The *Whitney topology* (*also called the fine topology*), that we denote by τ_W, is the topology on $\mathcal{H}(X)$ which has as local base at each $f \in \mathcal{H}(X)$ all sets of the following form, which we call *tubes*:

$$T(f,\varepsilon) := \{g \in \mathcal{H}(X):\ d(f(x),g(x)) < \varepsilon(x),\ \forall x \in X\},$$

where ε is a continuous function from X to the positive real numbers.

It is known that, having been given a topological characterisation, the Whitney topology τ_W is independent of the metric d, see [13]. The Whitney topology is admissible but not a group topology, in general.

- *Using compatible metrics*: *the limitation topology* (*see* [4,17]).

Let (X,d) stand for a metric space again. The *limitation topology* is the topology on $\mathcal{H}(X)$ which has as local base at each $f \in \mathcal{H}(X)$ all sets of the following form:

$$B(f,d) := \{g \in \mathcal{H}(X):\ \sup\{d(f(x),g(x)):\ x \in X\} < 1\}$$

as d runs through all metrics compatible with X [2,4].

In [2], it has been proven that the limitation topology on $\mathcal{H}(X)$ is an admissible group topology.

For metric spaces the fine uniform topology and the limitation topology agree. Therefore we can get the same topology using either metrics compatible with X or using continuous functions from X to the positive real numbers.

The Whitney topology τ_W is finer than the fine uniform topology τ_f, but the following is also true:

Theorem 2.1. *If τ_W is a group topology, then $\tau_W = \tau_f$.*

Proof. By our preceding remark, it suffices to prove that τ_f is finer than τ_W, and since we assume τ_W is a group topology we need only to show that any tube $T(f,\varepsilon)$ centered at the identity map f of X contains a τ_f-neighbourhood $B(f,\rho)$ of f for some compatible metric ρ. To this end, for any given x in X let

$$U_x := B_d\left(x, \frac{\varepsilon(x)}{4}\right) \cap \left\{y:\ \varepsilon(y) > \frac{\varepsilon(x)}{2}\right\}.$$

Then $\mathcal{U} = \{U_x:\ x \in X\}$ is an open cover of X. By a well-known theorem (e.g., see [9, IX.9.4]) there exists a compatible metric ρ such that the cover of ρ-balls of radius 1, $\{B_\rho(x,1):\ x \in X\}$, refines $\mathcal{U}$. This implies $B(f,\rho) \subseteq T(f,\varepsilon)$: indeed let $g \in B(f,\rho)$. Then, for any given $z \in X$, we have $\rho(z,g(z)) < 1$. Therefore, for some x in X, both z and $g(z)$ belong to the open set U_x. Thus, $d(z,g(z)) < \frac{\varepsilon(x)}{2}$. Next, $z \in U_x$ yields $\varepsilon(z) > \frac{\varepsilon(x)}{2}$. So, $d(z,g(z)) < \varepsilon(z)$ for each z in X, or equivalently $g \in T(f,\varepsilon)$. This completes the proof. □

3. The homeomorphism group $\mathcal{H}(\mathbb{R} \times \mathbb{Q})$

Now we look into $\mathcal{H}(\mathbb{R} \times \mathbb{Q})$. The study of a complex object as $\mathcal{H}(\mathbb{R} \times \mathbb{Q})$ is considerably simplified by splitting any self-homeomorphism of $\mathbb{R} \times \mathbb{Q}$ into its two natural halves. The study of the two halves separately allows us to acquire their own features and their interplay.

We start with a result of J. de Groot and T. Nishiura (Example 3.3.6 in [12]) and give a proof since it is very elementary.

Example 3.1. $\mathbb{R} \times \mathbb{Q}$ is not rim-compact.

We show that $\mathbb{R}\times\mathbb{Q}$ is not rim-compact when both $\mathbb{R}$ and $\mathbb{Q}$ carry the Euclidean metric. To prove this we show that the boundary of any non-empty bounded open subset of $\mathbb{R}\times\mathbb{Q}$ is not compact. Let A stand for a non-empty bounded open subset of $\mathbb{R}\times\mathbb{Q}$. Then, the boundary of A, F_rA, having a continuous image that is not compact, cannot be compact. To show this, let q_1, q_2 be two distinct rational numbers such that, as q runs through $[q_1,q_2]\cap\mathbb{Q}$, then A shares with every horizontal line $H_q=\mathbb{R}\times\{q\}$ a non-empty subset A_q, which is obviously bounded. Both the infimum and the supremum of A_q in H_q are boundary points of A and project just to q. Thus, $p_2(F_rA)$, the projection of $F_r(A)$ to $\mathbb{Q}$, contains the subspace $[q_1,q_2]\cap\mathbb{Q}$ of $\mathbb{Q}$, that is closed but not compact. Hence, $p_2(F_rA)$ is not compact and, consequently, from continuity of p_2, F_rA cannot be compact.

After splitting an arbitrary self-homeomorphism F in its two natural halves, $p_1\circ F$, $p_2\circ F$, where p_1, p_2 are the usual projections of $\mathbb{R}\times\mathbb{Q}$ over $\mathbb{R}$ and $\mathbb{Q}$ respectively, we focus on the second half $p_2\circ F$. Note the two following obvious facts: The components of $\mathbb{R}\times\mathbb{Q}$ are the subsets of the type $\mathbb{R}\times\{q\}$, as q runs through $\mathbb{Q}$, and furthermore every homeomorphism of $\mathbb{R}\times\mathbb{Q}$ takes components to components. Consequently, for any given q in $\mathbb{Q}$, the following occurs:

$$p_2\circ F(x,q)=p_2\circ F(x',q),\quad \forall x,x'\in\mathbb{R}.$$

This means that $p_2\circ F$ is independent of the point x in $\mathbb{R}$. This feature of $p_2\circ F$ allows us to define a map f_2 from $\mathbb{Q}$ to itself by the rule:

$$f_2: q\in\mathbb{Q}\to p_2\circ F(0,q)\in\mathbb{Q}. \tag{$*$}$$

Accordingly, it seems natural to identify the self-homeomorphism F with the pair (f_1,f_2), where $f_1=p_1\circ F:\mathbb{R}\times\mathbb{Q}\to\mathbb{R}$ and $f_2:\mathbb{Q}\to\mathbb{Q}$ is determined by $p_2\circ F$ as in $(*)$. Of course, both f_1, f_2 are continuous. The identity map of $\mathbb{R}\times\mathbb{Q}$ identifies with the pair $(p_1, i_{\mathbb{Q}})$, where p_1 is again the usual projection of $\mathbb{R}\times\mathbb{Q}$ on $\mathbb{R}$ and $i_{\mathbb{Q}}$ is the identity map of $\mathbb{Q}$. Next, if F identifies with (f_1,f_2) and G with (g_1,g_2), then their composition $G\circ F$ identifies with the pair (h_1,h_2) where:

$$h_1(x,q)=g_1\big(f_1(x,q),f_2(q)\big),\quad \forall(x,q)\in\mathbb{R}\times\mathbb{Q}$$

and

$$h_2(q)=g_2\big(f_2(q)\big),\quad \forall q\in\mathbb{Q}.$$

Hence, if the inverse homeomorphism F^{-1} of F identifies with (g_1,g_2), then:

$$g_1\big(f_1(x,q),f_2(q)\big)=x,\quad \forall(x,q)\in\mathbb{R}\times\mathbb{Q}$$

and

$$g_2\big(f_2(q)\big)=q,\quad \forall q\in\mathbb{Q}.$$

This implies $g_2=f_2^{-1}$. Thus, f_2 is in turn a homeomorphism of $\mathbb{Q}$ to itself whenever F is a homeomorphism of $R\times\mathbb{Q}$ to itself.

The above identification leads to a natural one-to-one correspondence from $\mathcal{H}(\mathbb{R}\times\mathbb{Q})$ into $C(\mathbb{R}\times\mathbb{Q},\mathbb{R})\times\mathcal{H}(\mathbb{Q})$, where $C(\mathbb{R}\times\mathbb{Q},\mathbb{R})$ is the set of all continuous functions from $\mathbb{R}\times\mathbb{Q}$ to the reals. In the proof of Theorem 1.1 we will have topologies on each of these three function spaces which make this one-to-one correspondence a topological embedding.

4. Fine group topologies

Let X stand for a metrisable space. In order to construct admissible group topologies on $\mathcal{H}(X)$ and its subgroups, we loosely follow the presentation in [2] of the fine uniform topology as the limitation topology. We introduce first the notion of a fine uniform topology associated with a class of metrics, then the notion of a fine group topology associated with a class of metrics. As a matter of fact, the least element in $\mathcal{L}_H(\mathbb{R}\times\mathbb{Q})$ is achieved as a fine group topology.

We start by considering *a way to produce new metrics from old ones*. Given any self-homeomorphism h of X and any metric d compatible with X, we define a new metric d_h on X by the following formula:

$$d_h(x,y):=d\big(h(x),h(y)\big),\quad \forall x,y\in X. \tag{$**$}$$

Actually, the metric d_h is compatible with X, whenever d is compatible with X and h is a homeomorphism from X to itself.

Now, let $\mathcal{D}(X)$ be a class of metrics compatible with X and $\mathcal{G}(X)$ a subgroup of $\mathcal{H}(X)$. We will refer to the uniform topology induced on $\mathcal{G}(X)$ by the supremum of the uniformities on X associated with metrics in $\mathcal{D}(X)$ as the *fine uniform topology on $\mathcal{G}(X)$ associated with (or generated by) $\mathcal{D}(X)$* and we will denote it as $\tau_{\mathcal{D},\mathcal{G}}$. Of course, the fine uniform topology τ_f is then generated by the full homeomorphism group $\mathcal{H}(X)$ and by the class of all metrics compatible with X.

It is easy to show that at any $f\in\mathcal{G}(X)$ the topology $\tau_{\mathcal{D},\mathcal{G}}$ admits as subbasic open neighbourhoods all sets of the form:

$$B(f,d,n):=\left\{g\in\mathcal{G}(X):\ \sup\big\{d\big(f(x),g(x)\big):\ x\in X\big\}<\frac{1}{n}\right\}$$

as d runs through $\mathcal{D}(X)$.

We say that a class $\mathcal{D}(X)$ is *invariant under the group* $\mathcal{G}(X)$ or is $\mathcal{G}(X)$*-invariant* if, whenever $d \in \mathcal{D}(X)$ and $h \in \mathcal{G}(X)$ then $d_h \in \mathcal{D}(X)$, where d_h is the metric on X previously defined by $(**)$.

Note that if d is a totally bounded metric compatible with X, and $h \in \mathcal{H}(X)$ then d_h is totally bounded and compatible with X. Hence if $\mathcal{D}(X)$ is the class of all totally bounded metrics compatible with the topology on X, then $\mathcal{D}(X)$ is invariant under $\mathcal{G}(X) = \mathcal{H}(X)$.

In our main theorem, $\mathcal{G}(X)$ will be the full homeomorphism group $\mathcal{G}(X) = \mathcal{H}(X)$, but the next results is no more difficult to prove for any $\mathcal{G}(X)$.

Theorem 4.1. *If $\mathcal{D}(X)$ is $\mathcal{G}(X)$-invariant, then the fine uniform topology $\tau_{\mathcal{D},\mathcal{G}}$ is a group topology on $\mathcal{G}(X)$.*

Proof. Assume $f \in \mathcal{G}(X)$ and $d \in \mathcal{D}(X)$. Then the inverse f^{-1} of f is in $\mathcal{G}(X)$ and, consequently, by the $\mathcal{G}(X)$-invariance of $\mathcal{D}(X)$, $d_{f^{-1}}$ is in $\mathcal{D}(X)$, too. We use the standard notation in group theory that if x^{-1} denotes the inverse element of x in G, then for $A \subset G$ we put $A^{-1} = \{a^{-1}\colon a \in A\}$. We check that:

$$B\bigl(f^{-1}, d, n\bigr)^{-1} = B(f, d_{f^{-1}}, n),$$

were

$$B(f, d, n) := \left\{ g \in \mathcal{G}(X)\colon \sup\bigl\{d\bigl(f(x), g(x)\bigr)\colon x \in X\bigr\} < \frac{1}{n} \right\}.$$

Thus

$$\begin{aligned} g \in B(f, d_{f^{-1}}, n) \quad &\Leftrightarrow \quad d_{f^{-1}}\bigl(f(x), g(x)\bigr) < \frac{1}{n} \quad \text{for all } x \in X \\ &\Leftrightarrow \quad d\bigl(f^{-1}\bigl(f(x)\bigr), f^{-1}\bigl(g(x)\bigr)\bigr) < \frac{1}{n} \quad \text{for all } x \in X \\ &\Leftrightarrow \quad d\bigl(x, f^{-1}\bigl(g(x)\bigr)\bigr) < \frac{1}{n} \quad \text{for all } x \in X. \end{aligned}$$

Thus by putting $y = g(x)$, it follows that

$$d\bigl(g^{-1}(y), f^{-1}(y)\bigr) < \frac{1}{n} \quad \text{for all } y \in X \quad \Leftrightarrow \quad g^{-1} \in B\bigl(f^{-1}, d, n\bigr).$$

Thus the inverse function in $\mathcal{G}(X)$ is continuous with respect to the fine uniform topology $\tau_{\mathcal{D},G}$. An analogous proof will show that the product (i.e., composition of homeomorphisms) in $\mathcal{G}(X)$ is also $\tau_{\mathcal{D},G}$-continuous. $\square$

Every class of metrics $\mathcal{D}(X)$ admits as $\mathcal{G}(X)$-invariant enlargement the wider class $\{d_h\colon d \in \mathcal{D}(X),\ h \in \mathcal{G}(X)\}$, which is also the minimal $\mathcal{G}(X)$-invariant enlargement of $\mathcal{D}(X)$. The previous result enables us to define the fine uniform topology on $\mathcal{G}(X)$ generated by the minimal $\mathcal{G}(X)$-invariant enlargement of $\mathcal{D}(X)$ as the *fine group topology on $\mathcal{G}(X)$ generated by $\mathcal{D}(X)$*. All such topologies are admissible because they are uniform topologies.

The same group combined with different classes of metrics gives rise to different fine group topologies. The full homeomorphism group $\mathcal{H}(\mathbb{Q})$ of the rational number space $\mathbb{Q}$, equipped with the Euclidean topology, though admitting no least admissible topology, nevertheless it still supports the closed–open topology as the least admissible group topology, see [6]. It should be emphasised that if C is closed and A is open in $\mathbb{Q}$ and $C \subseteq A$, then there exists a clopen set E such that $C \subseteq E \subseteq A$. Thus, the sets like the following:

$$[E, E] := \bigl\{ f \in \mathcal{H}(\mathbb{Q})\colon f(E) \subseteq E \bigr\}$$

give arbitrarily small neighbourhoods at the identity map of $\mathbb{Q}$, as E runs through all clopen sets in $\mathbb{Q}$. This entails the coincidence of the closed–open topology with the clopen–open topology on $\mathcal{H}(\mathbb{Q})$. At the same time, the clopen–open topology on $\mathcal{H}(\mathbb{Q})$ is the uniform topology induced by the Čech uniformity of $\mathbb{Q}$, which is the finest totally bounded uniformity compatible with $\mathcal{H}(\mathbb{Q})$, see [6]. Consequently, the clopen–open topology on $\mathcal{H}(\mathbb{Q})$ can be reformulated as the supremum of all uniform topologies induced by totally bounded metrics compatible with $\mathbb{Q}$, or, in other words, is the same as the fine group topology induced on the full group $\mathcal{H}(\mathbb{Q})$ by the full class of totally bounded metrics on $\mathbb{Q}$.

On the other hand, by Theorem 2.1, the fine group topology induced on the full group $\mathcal{H}(\mathbb{Q})$ by the full class of metrics on $\mathbb{Q}$ coincides with the Whitney topology, since the Whitney topology is a group topology, as proved in [6]. But the clopen–open topology and the Whitney topology on $\mathcal{H}(\mathbb{Q})$ do not agree, because the latter is strictly finer than the former, see [6].

5. Main result

Using techniques of fine group topologies, we now prove our main result.

We also use the following two special properties. The first property is that any two non-empty clopen subspaces of $\mathbb{Q}$ are homeomorphic [10, 6.2A(d)], and the second property is that any two closed bounded intervals in $\mathbb{R}$ are homeomorphic.

We recall the notion of product metric on a product space. Let (X_1, d_1), (X_2, d_2) stand for two metric spaces. Then, their product $X_1 \times X_2$ can be given the *product metric* $d_1 \times d_2$, which is defined by:

$$d_1 \times d_2\big((x_1, x_2), (y_1, y_2)\big) := \max\big\{d_1(x_1, y_1), d_2(x_2, y_2)\big\}.$$

If we inject $\mathcal{H}(\mathbb{R}\times\mathbb{Q})$ into $C(\mathbb{R}\times\mathbb{Q})\times\mathcal{H}(\mathbb{Q})$ using the canonical identification described in Section 3, then the following holds:

Lemma 5.1. *Let $\{F_\lambda\}_\lambda$ be a net in $\mathcal{H}(\mathbb{R}\times\mathbb{Q})$ and identify each F_λ with the pair $(f_{1\lambda}, f_{2\lambda}) \in C(\mathbb{R}\times\mathbb{Q})\times\mathcal{H}(\mathbb{Q})$. Then $\{F_\lambda\}_\lambda$ converges to the identity I of $\mathbb{R}\times\mathbb{Q}$ in the uniform topology induced by $d_1 \times d_2$ on $\mathcal{H}(\mathbb{R}\times\mathbb{Q})$ iff the net $\{f_{1\lambda}\}_\lambda$ of $C(\mathbb{R}\times\mathbb{Q})$ converges uniformly in d_1 to p_1 and the net $\{f_{2\lambda}\}_\lambda$ of $\mathcal{H}(\mathbb{Q})$ converges uniformly in d_2 to the identity map of $\mathbb{Q}$.*

We denote by d_1 the stereographic metric on $\mathbb{R}$, which measures the distance between two points in $\mathbb{R}$ as the geodesic (shortest) distance of their images in the unit circle S^1 of the Euclidean plane by the inverse of the stereographic projection.

Recall the notation: $\mathcal{L}_H(\mathbb{R}\times\mathbb{Q})$ denotes the set of all admissible group topologies on $\mathcal{H}(\mathbb{R}\times\mathbb{Q})$ ordered by inclusion.

Proof of Theorem 1.1. We must show that the set $\mathcal{L}_{\mathcal{H}(\mathbb{R}\times\mathbb{Q})}$ has a least element. This least element (i.e., topology) is defined as follows. Let $X = \mathbb{R}\times\mathbb{Q}$, $\mathcal{G}(X) = \mathcal{H}(X)$, and let $\mathcal{D}(X)$ be the class of all metrics on $\mathbb{R}\times\mathbb{Q}$ of the type $d_1 \times d_2$, where d_1 is the stereographic metric on $\mathbb{R}$ and d_2 runs through all totally bounded metrics compatible with the usual topology on $\mathbb{Q}$. Let τ_{tb} denote the fine group topology on $\mathcal{H}(X) = \mathcal{H}(\mathbb{R}\times\mathbb{Q})$ generated by $\mathcal{D}(X)$. Thus τ_{tb} is of the form $\tau_{\mathcal{D},\mathcal{G}}$. As we noted, this particular $\mathcal{D}(X)$ is $\mathcal{H}(X)$-invariant so we know that τ_{tb} is a group topology on $\mathcal{G}(X) = \mathcal{H}(\mathbb{R}\times\mathbb{Q})$ by an application of Theorem 4.1. Also τ_{tb} is admissible because it is defined from uniformities.

We now show that τ_{tb} is the least element in $\mathcal{L}_{\mathcal{H}(\mathbb{R}\times\mathbb{Q})}$. Suppose that τ is an admissible group topology on $\mathcal{H}(\mathbb{R}\times\mathbb{Q})$. Since both τ and τ_{tb} are group topologies on $\mathcal{H}(\mathbb{R}\times\mathbb{Q})$, to show that $\tau_{tb} \subseteq \tau$ it suffices to show that every τ-neighbourhood of I, the identity map on $\mathbb{R}\times\mathbb{Q}$, contains a τ_{tb}-neighbourhood of I. If this is not the case, then there exists a net $\{F_\lambda\}_{\lambda\in\wedge}$ in $\mathcal{H}(\mathbb{R}\times\mathbb{Q})$ which converges to I in τ but not in τ_{tb}.

Now we use the natural identification of F with (f_1, f_2) and Lemma 5.1. We have that τ_{tb} is the minimal full-invariant enlargement of the class of the metrics $d_1 \times d_2$, as described above, and $\mathcal{H}(\mathbb{Q})$ has the clopen–open topology which is the fine group topology generated by all totally bounded metrics compatible with $\mathbb{Q}$. The topology on $C(\mathbb{R}\times\mathbb{Q},\mathbb{R})$ is the uniform topology generated by the metric d_1 in $\mathbb{R}$. Since the net $\{F_\lambda\}_{\lambda\in\wedge}$ does not converge to I, and I is identified with $(p_1, i_\mathbb{Q})$, then the net $\{f_{1\lambda}\}_{\lambda\in\wedge}$ does not uniformly converge in d_1 to p_1, or the net $\{f_{2\lambda}\}_{\lambda\in\wedge}$ does not converge to the identity of $\mathbb{Q}$ in the clopen–open topology. Thus we have two cases to consider.

The case in which $\{f_{2\lambda}\}_{\lambda\in\wedge}$ does not converge to $i_\mathbb{Q}$ in the clopen–open topology includes some ideas that arise in the other case. For this reason, it is more convenient to discuss it first. In such a case we would find a clopen set E in $\mathbb{Q}$ and a cofinal subset $\wedge^*$ in $\wedge$ so that $f_{2\lambda}$ is outside $[E, E]$ whenever $\lambda \in \wedge^*$. Then, for each $\lambda \in \wedge^*$ we could select in E a point q_λ whose image $f_{2\lambda}(q_\lambda)$ does not belong to E. Let $x \in \mathbb{R}$ be arbitrary and pick $q \in E$. Then since τ is admissible the evaluation function at $(I, (x, q))$ is continuous. It is possible to select a τ-neighbourhood $\mathcal{U}$ of I, a neighbourhood W of x in $\mathbb{R}$ and a clopen set E_1 in $\mathbb{Q}$, containing q and strictly contained in E so that whenever F is in $\mathcal{U}$ then $F(W \times E_1) \subseteq \mathbb{R}\times E$. Of course $F_\lambda(W \times E_1) \subseteq \mathbb{R}\times E$ eventually. As a consequence, every q_λ belongs to $E \setminus E_1$. Now we construct a self-homeomorphism H of $\mathbb{R}\times\mathbb{Q}$ such that $H \circ F_\lambda \circ H^{-1}(W \times E_1) \nsubseteq \mathbb{R}\times E$ frequently. This would be a contradiction since τ is a group topology, and therefore every inner automorphism is an autohomeomorphism. To construct such an H, we first produce a self-homeomorphism h of $\mathbb{Q}$ choosing a homeomorphism g which identifies $E \setminus E_1$ with E_1 (since both non-empty clopen sets) and then gluing g and its inverse g^{-1} with the restriction of the identity to $\mathbb{Q}\setminus E$. Finally, we obtain the result by letting H be as (p_1, h), so getting $p_2 \circ H \circ F_\lambda \circ H^{-1}(x, h(q_\lambda))$ outside of E frequently, thus yielding $H \circ F_\lambda \circ H^{-1}(W \times E) \nsubseteq \mathbb{R}\times E$ frequently. This contradiction completes the proof of this case. For the remainder of the proof we may assume that $\{f_{2\lambda}\}_{\lambda\in\wedge}$ converge to $i_\mathbb{Q}$ in the clopen–open topology on $\mathcal{H}(\mathbb{Q})$.

Now we consider the other case. We assume that $\{f_{1\lambda}\}_{\lambda\in\wedge}$ does not converge uniformly in d_1 to p_1. Thus there would exist a positive real number ε and (x_λ, q_λ) in $\mathbb{R}\times\mathbb{Q}$ so that $d_1(f_{1\lambda}(x_\lambda, q_\lambda), x_\lambda) \geqslant \varepsilon$ frequently. Since d_1 is the stereographic metric on $\mathbb{R}$, the two nets $\{f_{1\lambda}(x_\lambda, q_\lambda)\}_\lambda$ and $\{x_\lambda\}_\lambda$ cannot both diverge to infinity at the same time, hence at least one of them must have an accumulation point; so by passing to a cofinal subset we may assume that one of these two nets converges. First we assume that $\{f_{1\lambda}(x_\lambda, q_\lambda)\}$ converges to a point x in $\mathbb{R}$. In this case, we could find a neighbourhood around x in $\mathbb{R}$, $[x - 2\eta, x + 2\eta]$, $\eta > 0$, containing no x_λ at all. Moreover, since the net of rational numbers $\{q_\lambda\}$ has to accumulate in the Stone–Čech compactification $\beta(\mathbb{Q})$ of $\mathbb{Q}$ and the latter admits a base of clopen sets, we could select in $\mathbb{Q}$ a clopen set E supposed without any loss of generality containing q_λ eventually. Let $q \in E$. By the admissibility of τ and the continuity of the evaluation function at $(I, (x, q))$ we may select a τ-neighbourhood $\mathcal{U}$ of I, a clopen set E_1 contained in E and a neighbourhood $K_1 = [x - \delta, x + \delta]$ of x, with $\delta < \eta$, corresponding to the neighbourhood $K = [x - \eta, x + \eta]$ of x so that whenever F belongs to $\mathcal{U}$ then $F(K_1 \times E_1) \subseteq K \times E$. To get the contradiction, we construct a self-homeomorphism H of $\mathbb{R}\times\mathbb{Q}$ as product $h_1 \times h_2$ of a pair h_1, h_2, where h_1 is in $\mathcal{H}(\mathbb{R})$ and h_2 in $\mathcal{H}(\mathbb{Q})$, namely $H(x, q) := (h_1(x), h_2(q))$. A homeomorphism h_1 in $\mathcal{H}(\mathbb{R})$ would be assigned through the homeomorphical stretch of $K_1 = [x - \delta, x + \delta]$ over $K = [x - \eta, x + \eta]$, the homeomorphical shrinking of $[x-2\eta, x-\delta]$ to $[x-2\eta, x-\eta]$ and of $[x+\delta, x+2\eta]$ to $[x+\eta, x+2\eta]$ and finally leaving all other points fixed. Furthermore, a homeomorphism h_2 in $\mathcal{H}(\mathbb{Q})$ would be obtained by gluing a homeomorphism which takes E_1 to E with a homeomorphism that takes $Q \setminus E_1$ to $Q \setminus E$. As noted in the previous case, in this case we are assuming that the net $\{f_{2\lambda}\}$ converge to the identity $i_{\mathbb{Q}}$ in the clopen–open topology. Hence $f_{2\lambda}(E) \subseteq E$ eventually. Hence, $f_{2\lambda}(q_\lambda)$ should belong to E eventually. Then we could choose in $K_1 \times E_1$ pairs $(\hat{x}_\lambda, \hat{q}_\lambda)$ so that $h_1(\hat{x}_\lambda) = f_{1\lambda}(x_\lambda, q_\lambda)$ and $h_2(\hat{q}_\lambda) = f_{2\lambda}(q_\lambda)$, thus yielding $p_1 \circ H^{-1} \circ F_\lambda^{-1} \circ H(\hat{x}_\lambda, \hat{q}_\lambda) = x_\lambda$ eventually without x_λ in K; which is a contradiction.

If the net $\{f_{1\lambda}(x_\lambda, q_\lambda)\}$ does not accumulate in $\mathbb{R}$, then a similar argument applies to the net $\{x_\lambda\}$ by taking $h_1(x) = x$, $h_2(\hat{q}_\lambda) = q_\lambda$ and choosing λ so that $H^{-1} \circ F_\lambda \circ H \in \mathcal{U}$. This completes the proof. □

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